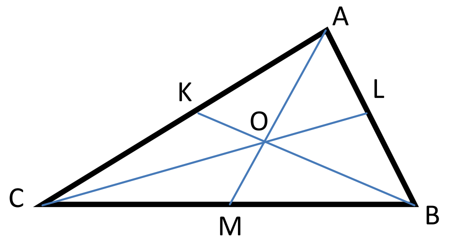
Form 10

1. Prove that the fraction $\frac{1340n+3}{2010n+4}$ can not be reduced for any whole positive number n.

Solution: for a fraction to be reducible its numerator and denominator should be divisible by the same number, that is greater than 1. But in that case the difference between numerator and denominator 670n+1 should also be divisible by the same number. Also the following difference (between numerator and the previous difference) $1340n + 3 - 670n - 1 = 670n + 2$ should be divisable by the same number. But $670n + 1$ and $670n + 2$ can not be divisable by any number greater than one at the same time. So the given fraction can not be reduced.

2. Let two medians of acute triangle ABC intersect in a right angle. Prove that, it is possible to construct a right angled triangle using all three medians of triangle ABC.



Solution: AL=OL as radii of the circle that is circumscribed around the right angled triangle. State that $CL = 3OL$. Using $AO^2 + BO^2 = (2OL)^2$ and substitution we can show that $\frac{4}{9}AM^2 + \frac{4}{9}BK^2 = \frac{4}{9}CL^2$. From that we deduce that $AM^2 + BK^2 = CL^2$ and that is the sufficient condition for constructing right angled triangle.

3. The quadratic equation $x^2 + ax + b = 0$ has two solutions x_1 and x_2. Find an equation such that it would have three solutions $x_1 + x_2$; $-x_1$; $-x_2$ and in the written form of the equation number a and b could be used, but numbers x_1 and x_2 can not be used.

Solution: Let us construct 2 equations. The first equation will have one root and it will be $x_1 + x_2 = -a$; the equation then is $(x + a) = 0$. The second will have two roots and their sum will be $-x_1 - x_2 = a$, but multiplication $x_1x_2 = b$; the equation then is $x^2 - ax + b = 0$. When we multiply both equations we get $x^3 + x(b - a^2) + ab = 0$.

4. Solve the inequality$|x - 1| < ax - a$, if a is given real number.

Solution: If $x > 1$ then $x - 1 < a(x - 1)$ and $(x - 1)(1 - a) < 0$ and for $a > 1$ the answer is $x > 1$. For $a \leq 1$ there are no solution.

If $x < 1$ then $1 - x < a(x - 1)$ and $(a + 1)(x - 1) > 0$ and $a + 1 < 0$, $a < -1$ the answer is $x < 1$. For $a \geq -1$ there are no solution.

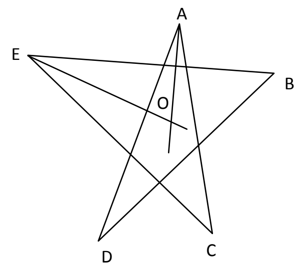
5. Consider n houses built in a circle, where each house has a number from 1 to n. For Christmas each of the n houses was given to one of the n dwarfs. To make it more fun, if the dwarf was assigned to the house with number k, it went to live in the house with number k^2 (counting from the first house and continuing to count for several circles if necessary). Prove that if n is divisible by 4, then there are no lonely dwarfs.

Note: After settling in each house can be inhabited by as many dwarfs as necessary. Lonely dwarfs are dwarfs that are living in the house alone.

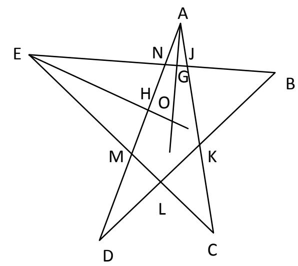
Solution: Let us consider the reminders $(2+p)^2$ and $(2n-p)^2$ for $p = 1, 2, \ldots., 2n-1$ when dividing by n. Expand the parentheses and see that both reminders when dividing by p^2 are the same, so the according dwarfs will settle in the same house. Dwarfs who received houses with numbers $2n$ and $4n$ will settle in the n-the house.

Form 11

1. Let ABCDE be five-sided star (the length of the sides are not necessarily equal). It is given that all the angles „at the vertices" are the same (for example $\angle ADB = \angle EBD$). Prove that the bisectors of any two consecutive vertex angles intersect in $72°$ angle. (In example angle $< EOA$ have to be $72°$)



Solution: Let us prove that all triangles ΔAJN, ΔBKJ, ΔCKL, ΔDLM, ΔEMN are isosceles. $\angle ENM = \angle ANJ$ as the vertical angles, but as the vertex angles A; B; C; D; E are also equal, then $\angle ENM = \angle ANJ = \angle BKJ$ continuing further around the circle , then using that $\angle AJN = \angle BJK$ and $\angle CLK = \angle DLM$ we deduce:



$\angle ENM = \angle ANJ = \angle BKJ = \angle CKL = \angle DML = \angle EMN.$

From the first and the last we can deduce that ΔENM is isosceles, similarly all triangles ΔAJN, ΔBKJ, ΔCKL, ΔDLM, are isosceles. All the angles in the pentagon JKLMN are then congruent and equal to $108°$ (for example $\angle HNJ = 108°$).

As bisector in an isosceles that is drawn from the vertex in between the equal sides is also the height, then in the quadrilateral HNGO we can calculate $\angle AOE = 360° - 108° - 90° - 90° = 72°$. The same refers to any other consecutive bisectors.

2. Find the polynomial P with degree 3, such that $P(1) = 3, P(2) = 0, P(3) = 1$.

Solution: As $P(2) = 0$, then let us find the polynomial P(x) in the form $P(x) = (x-2)(x^2 + ax + b)$. We can calculate the value of the second set of parentheses if we know the value of the first parentheses at a particular point and it is P*(1)=– 3 and P*(3)=1. Let us find P**(1)= –4 and P**(3)=0, and then we will use that P*(x)=P**(x)+1.

We can see that $P^{**} = (x-3)(x+d)$, where $(1-3)(1+d) = -4$.

We get $d = 1$; $P^{**} = (x-3)(x+1)$;

$P^{*} = (x-3)(x+1) + 1$;

$P(x) = (x-2)((x-3)(x+1)+1)$ and by expanding

$P(X) = P(x) = x^3 - 4x^2 + 2x + 4$.

By checking the answer we deduce that this is the correct solution.

It is possible to solve the task by creating system of equations P(x) or P*(x).

3. Function $g(x)$ is defined for real x as $g(x) = |x|$, if x is a whole number and as $g(x) = -\frac{1}{x}$, if x is not a whole number. Solve the inequality $g(x-1) < ax - a$, where a is given real positive number.

Solution:

a) If (x-1) is whole number, then $|x-1| < a(x-1)$ and the solution is as for Grade 10 only in whole numbers. Thus $x > 1$ if $a > 1$.

b) If (x-1) is not a whole number, then $\frac{1}{1-x} + a(1-x) < 0$

$\frac{1+a(1-x)^2}{1-x} < 0$. As $a > 0$, then the numerator is positive and the answer is all $1-x < 0$ or $x > 1$, where x is not whole number.

Answer: If $a > 1$, then $x > 1$. If $0 < a < 1$, then all not whole numbers $x > 1$, if $a = 1$ then there are no solutions.

4. Given that $a = (\sqrt{2})^x$. For which rational values of *a*, rational solution *x* exists?

Solution: Let us modify: $\left(\frac{k}{l}\right)^2 = 2^{\frac{p}{q}}$, where *p*, *k* – whole numbers, but *q*, *l* – whole positive numbers. And in fact *k* and *l* (and accordingly *p* and *q*) does not have any other divisor then 1. (It is possible that $k = 0$ or $p = 0$) .

Using the modification we get: $k^{2q} = l^{2q}2^p$.

If $l^{2q} = 1$, then $k = 2$ and $2q = p$ gives one of the solutions $a = 2^n$ (n = 1, 2, ...)

If $k^{2q} = 1$then $l = 2$ and $2q = p$ gives one more solution $a = \frac{1}{2^n}$ (n = 1, 2, ...)

If $p = 0$, $k = l$ and $a = 1$. Combing the answers we get that $a = 2^m$, where *m* is a whole number.

Let us prove that there are no other solutions. If *k* would consist of any other factor besides 2, then it would also be a factor of *l* and vice versa, but that is not possible as we chose *k* and *l* so that it wouldn't. As well *k* and *l* cannot consist of factor 2 at the same time. All other possibilities have already been discussed.

5. Consider *n* houses built in vertices of *n*-gone, where each house has a number from 1 to *n*. For Christmas each of the *n* houses was given to one of the *n* dwarfs. To make it more fun, if the dwarf was assigned to the house with number *k*, it went to live in the house with number k^2 (counting from the first house and continuing to count for several circles if necessary). Prove that:

1) if there is an odd number of dwarfs and there is exactly one lonely dwarf, it lives in the last house. 2) If there are exactly two lonely dwarfs, then they live as far from each other as possible.

Notes: After settling in each house can be inhabited by as many dwarfs as necessary. Lonely dwarfs are dwarfs that are living in the house alone.

Solution: If $n = 2k + 1$, then there will be a pair of dwarf $n + 1$ (or if counting in a circle – the first) and $n - 1$ .... $n + k$ and $n - k$. The only odd dwarf will be the n'th. It is not said that it will be lonely (for example in the case of $n = 9$), but, if there will be some one lonely, then it will be the n'th dwarf.

If $n = 4k + 2$, then similarly the extra ones will be dwarfs with numbers $2k + 1$ and $4k + 2$. The reminder for the first one will be $2k + 1$ as $(2k + 1)^2 = k(4k + 2) + 2k + 1$, but for the second one it will be zero. Thus if there will be two lonely dwarfs, then they will live at the ends of a diameter of the n-gons circumscribed circle.

Form 12

1. Prove the inequality $\tan\alpha + \cot\alpha \leq \tan^2\alpha + \cot^2\alpha$.

Solution:

$$\frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha} \leq \frac{\sin^2\alpha}{\cos^2\alpha} + \frac{\cos^2\alpha}{\sin^2\alpha}$$

$$\frac{1}{\cos\alpha\sin\alpha} \leq \frac{\sin^4\alpha + \cos^4\alpha}{\cos^2\alpha\sin^2\alpha}$$

$$\frac{1}{\cos\alpha\sin\alpha} \leq \frac{(\sin^2\alpha + \cos^2\alpha)^2 - 2\cos^2\alpha\sin^2\alpha}{\cos^2\alpha\sin^2\alpha}$$

$$\cos\alpha\sin\alpha \leq 1 - 2\cos^2\alpha\sin^2\alpha$$

$$2\cos\alpha\sin\alpha + 4\cos^2\alpha\sin^2\alpha \leq 2$$

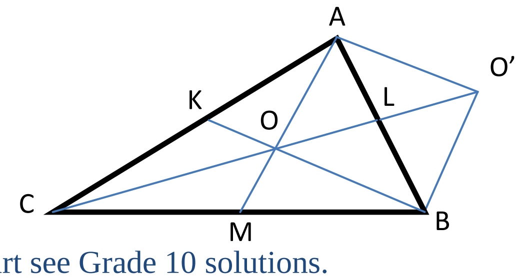
$$\sin 2\alpha + \sin^2 2\alpha \leq 2$$

As from the range of the function $\sin x$.

The second solution: substitute $\tan\alpha = x$ and solve $x + \frac{1}{x} \leq x^2 + \frac{1}{x^2}$. That is easy to be modified to $\frac{(x-1)^2(x^2+x+1)}{x^2} \geq 0$ where each of the factors is non-negative.

2. Consider a triangle that has medians with such length, that line segments with the same length could be sides of a right angled triangle. Prove that it is possible if and only if two of the medians are perpendicular.

Solution: $AM^2+KB^2=CL^2$ (*) from the given. $OL = \frac{1}{3}CL$; $AO = \frac{2}{3}AM$; $BO = \frac{2}{3}BK$. Substituting (*) we get $AO^2+BO^2=(2OL)^2$ (**). Let us extend OL so that O'L=OL. In the quadrilateral AOBO' diagonals are bisecting each other (AL=LB un OL=O'L) so it is parallelogram. Thus OA=BO'. And from (**) we can deduce that OBO' right angled triangle. And AOBO' is rectangle. So the angle AOB is right angle.



For the second part see Grade 10 solutions.

3. Let $x^3 - 3bx^2 + 4b^3 = 0$ be a cubic equation, with solutions x_1, x_2 and x_3. Given that $b > 0$. Prove that if $x_1 < 0$, then $x_2 = x_3$.

Solution: One of the solutions is $-b$. Factorize $(x+b)(x-2b)^2 = 0$, if $b > 0$, then there is one negative root and the positive root is doubled.

4. Prove that if $a^4 + b^4 + c^4 + d^4 + e^4$ is dividable with 5, than $a^4 - b^4$ also is divisible with 5.

Solution: Considering the 4th degrees reminders when dividing by 5, we can see that it could only be either 0 or 1 (The proof for that is necessary). That means: if the sum of five numbers is divisible by 5, then all the numbers should be divisible by 5, or they should all give the reminder 1. From that we can deduce the solution.

It is possible to consider the last digit and their possible sums.

5. Andris imagined 8 random numbers. With each question Valdis could find out the sum of any two of those numbers. After 12 questions he had asked about each of the numbers 3 times (in a combination with different numbers of course), but still insisted that he couldn't find out all the numbers that Andris had imagined. Is that possible?

Solution: Yes, it is possible. Let us locate the numbers 2 and -2 in eight cells of a cube so that on the first level there would be $\begin{matrix} +2 & -2 \\ -2 & +2 \end{matrix}$, but in the second $\begin{matrix} -2 & +2 \\ +2 & -2 \end{matrix}$. Let us ask four questions „parallel" to each of the coordinate axes (x, y and z). For each of those questions we receive an answer „zero". We would get the same answers if we would have changed the signs of the numbers to the opposite or we would multiply all eight numbers by any other number. All 12 questions meet the conditions of the task, as all the questions have been asked about different pairs and each of the numbers have been included in the question three times (in the direction of each of the axes). Nevertheless it is impossible to calculate the numbers.