Mathematics Solutions Science Olympiad 2012 English

Form 10

1. Assume that $x = a/b$ is a solution, where a and b are whole numbers and $b > 0$. This means that $2^{a/b} = 0$. Raise both sides to the power b, so that $2^a = 10^b = 2^b5^b$. It follows that $2^{a-b} = 5^b$. Since the right-hand side is larger than 5, it follows that $a > b$. The left-hand side is an even number while the right-hand side is an even number, which is impossible. $x = a/b$ cannot therefore be a solution.

   Another method is to compare the final digit on the two sides in the equality $2^a = 10^b$. On the left-hand side, the final digit is 2,4, 6 or 8, whereas on the right-hand side, the final digit is 0.

2. Extend the sides AB and CD so that they meet in the point E. The angle AED is then 30° and the triangles EAD and ECD are then both 30°-60°-90°-triangles. The length of the side CE is therefore 2cm, from which it follow that DE is 6 cm and thus that AD is $6/\sqrt{3}$ cm. According to Pythagoras, the distance AC is then equal to $\sqrt{(36/3 + 4^2)} = \sqrt{28}$ cm.

3. We can decide the matter by squaring both sides since, for the positive numbers a and b, $a \le b$ if and only if $a^2 \le b^2$.
   Let $a = 6 + \sqrt{26}$ and $b = \sqrt{123}$, so that $a^2 = 36 + 26 + 12\sqrt{26}$ and $b^2 = 123$.
   Then $a \le b$ if and only if $12\sqrt{26} \le 123 - 62 = 61$.
   Square both sides so that $a \le b$ if and only if $144 \cdot 26 \le 61^2 = 3721$; i.e. if and only if $3744 \le 3721$, which is false.
   This means that $a = 6 + \sqrt{26}$ is larger than $b = \sqrt{123}$.

4. a) Since 53 is not divisible by 7, the number 53 cannot be written in the form 7a, where a is a whole number. The number 43 is not divisible by 7 either, so 53 cannot be written in the form 7a + 10, where a is a whole number. Similarly, the numbers 33, 23, 13 and 3 are not divisible by 7, and 53 cannot therefore be written in the form 7a + 10b where a and b are non-negative whole numbers.

   b) In this case, $54 = 7\cdot2 + 10\cdot4$, $55 = 7\cdot5 + 10\cdot2$, $56 = 7\cdot8 + 10\cdot0$, $57 = 7\cdot1 + 10\cdot5$, $58 = 7\cdot4 + 10\cdot3$, $59 = 7\cdot7 + 10\cdot1$, $60 = 7\cdot0 + 10\cdot6$, $61 = 7\cdot3 + 10\cdot4$, $62 = 7\cdot6 + 10\cdot2$, $63 = 7\cdot9 + 10\cdot0$, $64 = 7\cdot2 + 10\cdot5$, $65 = 7\cdot5 + 10\cdot3$, $66 = 7\cdot8 + 10\cdot1$, $67 = 7\cdot1 + 10\cdot6$, $68 = 7\cdot4 + 10\cdot4$, $69 = 7\cdot7 + 10\cdot2$, $70 = 7\cdot10 + 10\cdot0$.

   c) If n is a whole number > 60, there is a positive whole number k such that $54 \le n - 7k \le 60$. According to the above, there exist non-negative whole numbers a and b such that $n - 7k = 7a + 10b$, therefore $n = 7(a + k) + 10b$.

5. According to the formula for a geometric series,

$$x^4 + x^3 + x^2 + x + 1 = (x^5 - 1)/(x - 1)$$

from which it follows that

$$(x^4 + x^3 + x^2 + x + 1)(x - 1) = (x^5 - 1),$$

which can also be seen by carrying out the binomial multiplication

$$(x^4 + x^3 + x^2 + x + 1)(x - 1).$$

If x is real number that solves the equation $x^4 + x^3 + x^2 + x + 1 = 0$, then it is evident that $x^5 = 1$, but $y = x^5$ is a continuously growing function, so the only real root of the equation $x^5 = 1$ is $x = 1$.
$x = 1$ is not however a solution of $x^4 + x^3 + x^2 + x + 1 = 0$.

An alternative approach is to first realise that a solution to the equation must be negative. If $-1 \le x \le 0$, then $x + 1 \ge 0$ and therefore
$x^4 + x^3 + x^2 + x + 1 = x^4 + x^2(x + 1) + x(x + 1) \ge x^4 > 0$. If $x < -1$, then
$x^4 + x^3 + x^2 + x + 1 = x^3(x + 1) + x(x + 1) + 1 > 0$.

Form 11

1. If a head appears, A will certainly win, since eventually (with probability 1) a tail will appear after a series of only heads. The only chance for B to win is therefore if the first two throws are both tails. The probability of this happening is 1/4, so that the probability of A winning is 3/4.

2. The line AB can be extended to a chord A´B´of the circle. The line CA´ intersects the circle in the point E and the line DA´ intersects the circle in the point F. The angle A´EB´ is the sum of the angles A´CB´ and EB´C, and the angle A´EB´ is therefore larger than the angle A´CB´, which in turn is larger than the angle ACB. One can reason in the same way on the other side of the line passing through A and B to show that the angle A´FB´ is larger than the angle ADB. The sum of the two angles A´EB´ and A´FB´ subtended at the circumference is 180 °, since they are each equal to half the corresponding angle subtended at the centre which together create a full circle. The conclusion is that the sum of the angles ACB and ADB is less than the sum of the angles A´EB´ and A´FB´ which is 180°.

3. The following is true:

$$(2a + 1)^2 - (2a - 1)^2 = 4a^2 + 4a + 1 - (4a^2 - 4a + 1) = 8a.$$

4. After some trial and error, it is evident that 53 may be a possible answer. Let us prove this. Since 53 is not divisible by 7,the number 53 cannot be written as in the form 7a where a is a whole number. The number 43 is not divisible by 7 either, so 53 cannot be written in the form 7a + 10, where a is a whole number. Similarly, the numbers 33, 23, 13 and 3 are not divisible by 7, and 53 cannot therefore be written in the form 7a + 10b where a and b are non-negative whole numbers.

   Further, $54 = 7\cdot2 + 10\cdot4$, $55 = 7\cdot5 + 10\cdot2$, $56 = 7\cdot8 + 10\cdot0$, $57 = 7\cdot1 + 10\cdot5$, $58 = 7\cdot4 + 10\cdot3$, $59 = 7\cdot7 + 10\cdot1$, $60 = 7\cdot0 + 10\cdot6$.

   If n is a whole number > 60, there is a positive whole number k such that $54 \le n - 7k \le 60$. According to the above, there exist non-negative whole numbers a and b such that $n - 7k = 7a + 10b$, therefore $n = 7(a + k) + 10b$.

The answer is thus 53.

5. Assume that S is sparse sub-set of T. There is a total of 23 pairs of elements from the set {1, 2, 3, 4, 5, 6, 7} and each pair occurs in not more than one set in S. Each set in S contains exactly 3 pairs. Thus S contains not more than 7 sets. An example of a sparse sub-set of T containing 7 sets is

   {{1, 2, 3},{1, 4, 5},{1, 6, 7},{2, 4, 6},{2, 5, 7},{3, 4, 7},{3, 5, 6}}

Form 12

1. After two throws, the possible combinations are *head*, *head*; *tail, head*; *tail, tail* and *head, tail,* each of which may occur with equal probability 1/4. If the combination *head, head* occurs, A will certainly win since eventually (with probability 1) a tail will appear after a series of only heads. If the combination *tail, head* occurs, B will win with a probability of 1/2 since he wins if a tail is thrown next and, as indicated above, he loses if a head is thrown next. If either of the combinations *tail, tail* or *head, tail* occurs, then eventually (with probability 1) a head will appear after a series of only tails. As indicated above, B will then win with a probability of 1/2. With a probability of 3/4 (i.e. if the first two throws are not *head, head*), there is thus a probability of 1/2 that B will win. The probability that B will win is thus 3/8 and the probability that A will win is 5/8.

2. Assume that $c > ab – a – b$ and that x_0, y_0 are whole numbers such that $ax_0 + by_0 = c$. If neither x_0 nor y_0 is negative, we are finished. Otherwise either $x_0 \leq -1$ or $y_0 \leq -1$. These two cases are completely analogous, so it is sufficient to consider the case when $x_0 \leq -1$. Then $ab – a – b < c = ax_0 + by_0 \leq -a + by_0$ and thus $ab – b < by_0$, which gives $a – 1 < y_0$, i.e. $y_0 \geq a$. It is also true that $x_1 = x_0 + b$, $y_1 = y_0 + a$, where $y_1 \geq 0$ is a solution. If $x_1 \geq 0$, we are finished. Assume that $x_1 \leq -1$. The same reasoning as above then gives that $y_1 \geq a$, so that we can consider the solution $x_2 = x_1 + b$, $y_2 = y_1 + a$ when $y_2 \geq 0$. Continuing in this manner leads finally to the solution x_n, y_n when $x_n \geq 0$ and $y_n \geq 0$.

3. The angle of incidence at the second reflection is 5 + 7 degrees, and at the third reflection 5 + 2·7 degrees. The beam is reflected inwards as long as the angle of incidence is less than 90°. The angle at the 13th reflection is 5 + 12.7 = 89°. At the 14th reflection, the beam is reflected outwards at an angle of 180 – (89 + 7) = 84°. The angle at the 15th reflection is 84 – 7 degrees and the angle at the 25th reflection is 84 – 11·7 degrees. The angle of reflection is then parallel to one side of the funnel and the beam therefore leaves the funnel. The answer is a total of 25 reflections.

4. Assume first that x_0, y_0 are whole numbers so that $x_0^2 – y_0^2 = a$. It is then necessary to show that a does not have the form $4n + 2$ (since an arbitrary whole number is either of the form $4n \pm 1$ or $4n + 2$).

If only one of x_0 and y_0 is even, then $x_0^2 - y_0^2 = a$ is odd, and if both x_0 and y_0 are even, then x_0^2 and y_0^2 are divisible by 4 and thus a is also divisible by 4.

Assume that $x_0 = 2m + 1$ and $y_0 = 2n + 1$. Both are odd.
Then $x_0^2 – y_0^2 = 4m^2 + 4m + 1 – (4n^2 + 4n + 1) = 4(m^2 + m - n^2 - n)$, so that a is also divisible by 4 in this case.

Assume now the opposite that a is odd or is divisible by 4. It is now necessary to show that there are whole numbers x_0, y_0 such that $x_0^2 – y_0^2 = a$. Assume first that $a = 2n + 1$ is odd.

Let $x_0 = n + 1$ and $y_0 = n$. In this case, $x_0^2 – y_0^2 = (n + 1)^2 - n^2 = 2n + 1 = a$. Assume that $a = 4b$, where b is an odd number. There exists then as above x_1, y_1 such that $x_1^2 – y_1^2 = b$. Let $x_0 = 2x_1$ and $y_0 = 2y_1$. Then it is true that $x_0^2 – y_0^2 = 4(x_1^2 - y_1^2) = 4b = a$. Finally, assume that $a = 4b$ where b is an even number. In that case, $a = 8c$. Let $x_0 = 2c + 1$ and $y_0 = 2c – 1$. Then it is true that $x_0^2 – y_0^2 = 4c^2 + 4c + 1 – (4c^2 – 4c + 1) = 8c = a$.

5 a) The series created by the final digits of the numbers in the series $a_1, a_2, a_3 \ldots$ is 1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2, 1, 3, 4, …… . This series thus repeats itself with a period of 12 and none of the final digits is 0. This is true of all the numbers in the series, and therefore there is no number in the series that is divisible by 10.

b) Consider the series $c_1, c_2, c_3 \ldots$ which is based on the three final digits of the numbers in the series $b_1, b_2, b_3 \ldots$, and consider all the pairs c_n, c_{n+1}. There are at the most 1000 possible values of c_n and thus 10^6 possible pairs. The pairs must therefore eventually repeat themselves. Let k be the smallest suffix so that the pair c_k, c_{k+1} repeats, and let $c_k = c_j$ and $c_{k+1} = c_{j+1}$. It must then be true that $k = 0$. If not, assume that $k > 0$ and assume that $c_k \leq c_{k+1}$. In that case, $c_{k-1} = c_{k+1} - c_k = c_{j+1} – c_j = c_{j-1}$ so that the pair c_{k-1}, c_k repeats, which is a contradiction. Assume that $c_k > c_{k+1}$. In that case, $c_{k-1} = c_{k+1} - c_k + 1000 = c_{j+1} – c_j + 1000 = c_{j-1}$ so that the pair c_k, c_{k+1} also repeats in this case, which is a contradiction. Thus $k = 0$ and consequently $c_0 = 0$ is repeated, which means that some term b_n is divisible by 1000.