# The 28th International Science Olympiad for Young Mathematicians, Physicists and Chemists

November 3, 2015

Mathematics - Form 10

**1.** Place the integers $4^{4^5}$, $4^{5^4}$, $4^{5^5}$, $5^{4^5}$, $5^{4^4}$ in an increasing order. Here $a^{b^c}$ denotes $a^{(b^c)}$.

**2.** Let $a$ and $b$ be positive real numbers such that $a+b$ and $ab$ are rational numbers. Do $a$ and $b$ necessarily have to be rational numbers too?

**3.** Let $a$ and $b$ be positive real numbers such that

$$4a + 12\sqrt{ab} - 4\sqrt{a} - 6\sqrt{b} + 9b = 8.$$

Find the value of

$$\frac{2\sqrt{a} + 3\sqrt{b} + 2015}{2015 - 4\sqrt{a} - 6\sqrt{b}}.$$

**4.** Consider the equation

$$x^4 + x^3 - x^2 - 2x - 2 = 0.$$

a) Show that there is at least one real number $x$ that satisfies the equation.

b) Find all real numbers $x$ that satisfy the equation.

**5.** Let $ABCD$ be a trapezoid such that $AB$ and $CD$ are parallel and

$$|DA| = |AB| = |BC| = \frac{1}{2}|DC| = 1.$$

Find the length of $AC$.

*Each full solution is worth 5 points. Just answers will not suffice to obtain full points, a thorough justification is always expected!*

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November 3, 2015

Mathematics - Form 11

**1.** An integer is a *perfect cube* if it is a cube of an integer.

a) Give an example of 6 positive integers $a$, $b$, $c$, $d$, $e$ and $f$, none of them perfect cubes, such that all of $ab$, $bc$, $cd$, $de$, $ef$ and $fa$ are perfect cubes.

b) Show that you cannot find 5 positive integers $a$, $b$, $c$, $d$, and $e$, none of them perfect cubes, such that all of $ab$, $bc$, $cd$, $de$, $ea$ are perfect cubes.

**2.** Is there an integer $a$ such that $a^2 = 201520156789$?

**3.** Find the number of pairs of positive integers $(a, b)$, such that $a + b < 100$ and

$$a + \frac{1}{b} = 13\left(b + \frac{1}{a}\right).$$

**4.** Show that all internal angles of an equilateral pentagon with an inscribed circle are equal.

**5.** Find the maximal and minimal possible number of different real-valued solutions to the equation

$$P(P(P(x))) = 0,$$

where $P$ is a second degree polynomial. Find with proof an example which has exactly the minimal possible number of solutions and an example which has the maximal possible number of solutions. *Note that $P(P(x))$ denotes repeated application. For example, if $P(x) = x^2 - 1$, then $P(P(x)) = (x^2 - 1)^2 - 1$.*

*Each full solution is worth 5 points. Just answers will not suffice to obtain full points, a thorough justification is always expected!*

# The 28th International Science Olympiad for Young Mathematicians, Physicists and Chemists

November 3, 2015

Mathematics - Form 12

**1.** Let $a \geq 5$ be an integer. Show that $a^2 + 16$ is not divisible by $a^2 - 16$.

**2.** James has 3 dice. He can erase all the 6 numbers written on them and write a new number on each side of each dice, using any real number as many times as he wants. Can he do it in such a way that when the three dice are rolled at the same time, all of the following hold:

1) the probability that the first shows a higher number than the second is strictly greater than $\frac{1}{2}$,
2) the probability that the second shows a greater number than the third is strictly greater than $\frac{1}{2}$,
3) the probability that the third dice will show a higher number than the first is also strictly greater than $\frac{1}{2}$?

**3.** Find all positive integer solutions $x, y$ to the following equation:

$$10^x + x = 10^y + y^2.$$

**4.** Let $ABC$ be a right triangle with $\angle ACB = 90°$ and let $D$ be a point on the side $AB$ different from both $A$ and $B$. Let points $E$ and $F$ be chosen on the sides $AC$ and $CB$ respectively, so that $AC \perp ED$ and $CB \perp FD$. Prove that $S_{ABC} \geq 4S_{EFD}$, where $S_{XYZ}$ denotes the area of the triangle $XYZ$.

**5.** A sequence $a_1, a_2, \ldots$ of positive real numbers is called harmonic if for all $n > 0$, $a_{n+1}$ is the harmonic mean of $a_n$ and $a_{n+2}$, i.e. if for all $n > 0$ we have:

$$a_{n+1} = \frac{2}{\frac{1}{a_n} + \frac{1}{a_{n+2}}}.$$

Let $a_1, a_2, \ldots$ be a harmonic sequence of positive real numbers with $a_1 = 1$ and $a_2 = \frac{1}{2}$. Find $a_{2015}$.

*Each full solution is worth 5 points. Just answers will not suffice to obtain full points, a thorough justification is always expected!*

# Problems and solutions

November 3, 2015

Mathematics - Form 10

**1.** Place the integers $4^{4^5}$, $4^{5^4}$, $4^{5^5}$, $5^{4^5}$, $5^{4^4}$ in an increasing order. Here $a^{b^c}$ denotes $a^{(b^c)}$.

**Solution.** The order is $5^{4^4} < 4^{5^4} < 4^{4^5} < 5^{4^5} < 4^{5^5}$. Firstly, note that $5^4 < 4^5$, and hence $4^{5^4} < 4^{4^5}$. Clearly $4^{4^5} < 5^{4^5}$. Now

$$5^{4^4} = (5^4)^{64} < (4^5)^{64} < (4^5)^{125} = 4^{5^4}.$$

Finally, note that

$$4^{5^5} = (4^5)^{625} > (5^4)^{625} > 5^{4^5}.$$

**2.** Let $a$ and $b$ be positive real numbers such that $a+b$ and $ab$ are rational numbers. Do $a$ and $b$ necessarily have to be rational numbers too?

**Solution.** No. For example, if $a = 2+\sqrt{2}$ and $b = 2-\sqrt{2}$, then $a+b=4$ and $ab=2$ are rational, but both $a$ and $b$ are irrational and positive.

**3.** Let $a$ and $b$ be positive real numbers such that

$$4a + 12\sqrt{ab} - 4\sqrt{a} - 6\sqrt{b} + 9b = 8.$$

Find the value of

$$\frac{2\sqrt{a}+3\sqrt{b}+2015}{2015-4\sqrt{a}-6\sqrt{b}}.$$

**Solution.** Denote $m = 2\sqrt{a}+3\sqrt{b}$. Then we can write the condition as:

$$m^2 - 2m = 8,$$

or

$$(m-4)(m+2) = 0.$$

Thus as $m > 0$ we get that $2\sqrt{a}+3\sqrt{b} = 4$. Hence we have

$$\frac{2\sqrt{a}+3\sqrt{b}+2015}{2015-4\sqrt{a}-6\sqrt{b}} = \frac{4+2015}{2015-8} = \frac{2019}{2007}.$$

**4.** Consider the equation

$$x^4 + x^3 - x^2 - 2x - 2 = 0.$$

a) Show that there is at least one real number $x$ that satisfies the equation.

b) Find all real numbers $x$ that satisfy the equation.

**Solution.** a) Calculating the values of the function $f(x) = x^4 + x^3 - x^2 - 2x - 2$ at $x = 1$ and $x = 2$, we find $f(1) = -2$ and $f(2) = 16 + 8 - 4 - 4 - 2 = 14$. As the function $f$ is negative at $x = 1$ and positive at $x = 2$, it must have at least one solution in the interval $(1, 2)$.

b) Adding and subtracting $x^2$ to the original equation, we find

$$x^4 + x^3 - x^2 - 2x - 2 = (x^4 + x^3 + x^2) + (-2x^2 - 2x - 2).$$

We can simplify this to

$$x^2(x^2 + x + 1) - 2(x^2 + x + 1) = (x^2 + x + 1)(x^2 - 2).$$

We can factor the last factor as

$$x^2 - 2 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right).$$

We have found that

$$x^4 + x^3 - x^2 - 2x - 2 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)(x^2 + x + 1).$$

The quadratic formula shows that $x^2 + x + 1$ has no real solutions. Therefore, the only real solutions are $x = \sqrt{2}$ and $x = -\sqrt{2}$.

**5.** Let $ABCD$ be a trapezoid such that $AB$ and $CD$ are parallel and $|DA| = |AB| = |BC| = \frac{1}{2}|DC| = 1$. Find the length of $AC$.

**Solution.** Let $M$ be the midpoint of the side $DC$. Then $AMCB$ is a parallelogram: indeed, $AB$ and $MC$ are both parallel and have equal length. Hence in particular $|AM| = |BC|$ and so $|MD| = |MA| = |MC|$. thus $DC$ is a diameter of circumcircle of triangle $CAD$ which tells us that $\angle DAC = 90°$ and so by the Pythagorean theorem we get

$$|AC| = \sqrt{|DC|^2 - |AD|^2} = \sqrt{3}.$$

# Problems and solutions

November 3, 2015

Mathematics - Form 11

**1.** An integer is a *perfect cube* if it is a cube of an integer.

a) Give an example of 6 positive integers $a$, $b$, $c$, $d$, $e$ and $f$, none of them perfect cubes, such that all of $ab$, $bc$, $cd$, $de$, $ef$ and $fa$ are perfect cubes.

b) Show that you cannot find 5 positive integers $a$, $b$, $c$, $d$, and $e$, none of them perfect cubes, such that all of $ab$, $bc$, $cd$, $de$, $ea$ are perfect cubes.

**Solution.** For the first part, we can take $a = c = e = 2$ and $b = d = f = 4$, as then $ab = bc = cd = de = ef = fa = 2^3$. For the second part we argue as follows: as $a$ is not a cube of a positive integer, it has some prime factor $p$ that appears with an exponent $3k + 1$ or $3k + 2$ for some non-negative $k$. Without loss of generality, suppose it appears as $3k + 1$, the other case being treated similarly. Now as $ab$ is a cube, it means that $b$ must have this same prime factor respectively to the power $3n + 2$ for some non-negative $n$. Continuing the same argument for $c, d, e$ we see that $e$ must contain this prime factor to the power $3m + 1$. This however means that $ea$ contains this prime factor to the power $3(k + m) + 2$ and hence is not a cube.

**2.** Is there an integer $a$ such that $a^2 = 201520156789$?

**Solution 1.** No. If $a$ was such an integer then it must be an odd number. So $a = 2k + 1$ for some integer $k$ and hence

$$(2k + 1)^2 = 4k^2 + 4k + 1 = 201520156789.$$

Subtracting one from both sides gives

$$4k(k + 1) = 201520156788.$$

Since exactly one of the consecutive integers is divisible be 2, we find that $4k(k + 1)$ is divisible by 8. On the other hand, 201520156788 is not divisible by 8 since the number formed by the last three digits is not divisible by 8. So there is no such $a$.

**Solution 2.** No. We just note that $448909^2 = 201519290281$ and $448910^2 = 201520188100$, hence $448909 < \sqrt{201520156789} < 448910$.

**3.** Find the number of pairs of positive integers $(a, b)$, such that $a + b < 100$ and

$$a + \frac{1}{b} = 13\left(b + \frac{1}{a}\right).$$

**Solution.** From the second condition we get

$$a - 13b = \frac{13}{a} - \frac{1}{b} = \frac{13b - a}{ab}.$$

Here left and right hand sides have different signs unless $a - 13b = 0$. Hence $a = 13b$ and by condition $a + b < 100$ we see that only $b = 1, \ldots, 7$ are suitable options. Easy check shows that all pairs $(13b, b)$, $b = 1, 2, \ldots, 7$ satisfy all the conditions, so there are 7 such pairs.

**4.** Show that all internal angles of an equilateral pentagon with an inscribed circle are equal.

**Solution.** Let $A_1A_2A_3A_4A_5$ be an equilateral pentagon and with its inscribed circle touching its sides $A_1A_2$, $A_2A_3$, $A_3A_4$, $A_4A_5$ and $A_5A_1$ respectively at points $P_1$, $P_2$, $P_3$, $P_4$ and $P_5$. Let us denote $a = |A_1A_2| = \ldots = |A_4A_5| = |A_5A_1|$. It is a known fact that the two tangents drawn from a point to a circle are of equal length. Let us denote $b_1 = |A_1P_1| = |P_5A_1|$, $b_2 = |A_2P_2| = |P_1A_2|$, $\ldots$,$b_5 = |A_5P_5| = |P_4A_5|$. We find

$$\begin{cases} b_1 + b_2 = a \\ b_2 + b_3 = a \\ b_3 + b_4 = a \\ b_4 + b_5 = a \\ b_5 + b_1 = a \end{cases}$$

By adding the 1st, 3rd and 5th equation and subtracting the 2nd and 4th equation, we see that $2b_1 = a$. It now follows that $b_1 = b_2 = b_3 = b_4 = b_5 = a/2$.

Let us denote the center point of the incircle as $O$. The triangles $A_1P_5O$ and $A_1P_1O$ are congruent since $|A_1P_5| = |A_1P_1|$, $|P_1O| = |P_5O|$ and they share the side $A_1O$. Therefore, $\angle P_5A_1O = \angle P_1A_1O$. In a similar manner, $\angle P_1A_1O = \angle P_1A_2O$ and $\angle P_1A_2O = \angle P_2A_2O$. We find $\angle A_5A_1A_2 = \angle P_5A_1O + \angle P_1A_1O = \angle P_1A_2O + \angle P_2A_2O = \angle A_1A_2A_3$. Analogously for the other angles. Therefore, all the internal angles of the equilateral pentagon are equal.

**5.** Find the maximal and minimal possible number of different real-valued solutions to the equation

$$P(P(P(x))) = 0,$$

where $P$ is a second degree polynomial. Find with proof an example which has exactly the minimal possible number of solutions and an example which has the maximal possible number of solutions. *Note that $P(P(x))$ denotes repeated application. For example, if $P(x) = x^2 - 1$, then $P(P(x)) = (x^2 - 1)^2 - 1$.*

**Solution.** The minimal number of solutions is 0 and the maximal number of solutions is 8.

If we choose $P(x) = x^2 + 2$, then there are no solutions to $P(P(P(x))) = 0$, as $P(y)$ sends every number $y = P(P(x))$ to a positive number.

Notice that on the other hand $P(P(P(x)))$ is a polynomial of degree 8 and thus does not have more than 8 zeros. Let us choose $P(x) = x^2 - 2$. We will show that the solutions to $P(x)$ are $\pm\sqrt{2 \pm \sqrt{2 \pm \sqrt{2}}}$. Let $c$ be one of such numbers. Then $P(c) = \pm\sqrt{2 \pm \sqrt{2}}$. Similarly $P(P(c)) = \pm\sqrt{2}$ and $P(P(P(c))) = 0$. So all such numbers $c$ are solutions.

There are 8 ways to choose the signs. Let us show that all result in different numbers. It is clear that the numbers $\sqrt{2 \pm \sqrt{2 \pm \sqrt{2}}}$ are all different from the numbers $-\sqrt{2 \pm \sqrt{2 \pm \sqrt{2}}}$ as the first set contains positive numbers, the second one negative numbers. It suffices to show that all the numbers $\sqrt{2 \pm \sqrt{2 \pm \sqrt{2}}}$ are different from each other. If the squares of numbers are different, then the numbers are different, so it suffices to show that $2 \pm \sqrt{2 \pm \sqrt{2}}$ are all different numbers. If the numbers are different subtracting a constant, they were different before also, so it suffices to show that $\pm\sqrt{2 \pm \sqrt{2}}$ are all different numbers.

As in above, the positive numbers $\sqrt{2 \pm \sqrt{2}}$ are all different from the negative numbers $-\sqrt{2 \pm \sqrt{2}}$. It suffices to prove that the numbers $\sqrt{2 \pm \sqrt{2}}$ are all different. Again, taking the square and subtracting two, it suffices to show that the numbers $\pm\sqrt{2}$ are all different. But this is obvious. So, $P(P(P(x)))$ has exactly 8 solutions, which is the maximal number.

# Problems and solutions

November 3, 2015

Mathematics - Form 12

**1.** Let $a \geq 5$ be an integer. Show that $a^2 + 16$ is not divisible by $a^2 - 16$.

**Solution.** Assume on the contrary that $a^2 + 16$ is divisible by $a^2 - 16$. Then also the difference $32 = (a^2 + 16) - (a^2 - 16)$ must be divisible by $a^2 - 16$, which implies $a^2 - 16 \leq 32$. From this, we find $a^2 \leq 48 < 49$, giving $a < 7$. Checking the numbers $a = 5$ and $a = 6$ proves the claim.

**2.** James has 3 dice. He can erase all the 6 numbers written on them and write a new number on each side of each dice, using any real number as many times as he wants. Can he do it in such a way that when the three dice are rolled at the same time, all of the following hold:

1) the probability that the first shows a higher number than the second is strictly greater than $\frac{1}{2}$,
2) the probability that the second shows a greater number than the third is strictly greater than $\frac{1}{2}$,
3) the probability that the third dice will show a higher number than the first is also strictly greater than $\frac{1}{2}$?

**Solution.** Yes. For example, if James writes numbers 5, 5, 2, 2, 2 and 2 to the sides of the first dice, numbers 4, 4, 4, 4, 1 and 1 to the sides of the second dice and numbers 3, 3, 3, 3, 3 and 3 to the sides of the third dice, then these three probabilities are $\frac{5}{9}$, $\frac{2}{3}$ and $\frac{2}{3}$, respectively.

**3.** Find all positive integer solutions $x, y$ to the following equation:

$$10^x + x = 10^y + y^2.$$

**Solution.** Note that the pair $x = y = 1$ is a solutions. We now show that are no other such pairs in positive integers.

It is easy to see that if $x = 1$, then we must also have $y = 1$, and vice versa. From now on we assume $x, y \geq 2$. It is clear that we must have $x > y$. Therefore, $y \leq x - 1$ and it follows that

$$10^x + x = 10^y + y^2 \leq 10^{x-1} + (x-1)^2$$

Rearranging terms, we find

$$9 \cdot 10^{x-1} \leq (x-1)^2 - x.$$

We will show that

$$9 \cdot 10^{x-1} > (x-1)^2 - x.$$

The contradiction shows that there are no more solutions.

The inequality

$$2^x > x$$

holds, since $x \geq 2$ and doubling a number clearly increases it more than increasing it by one. Using this inequality, we find

$$9 \cdot 10^{x-1} > 9 \cdot 4^{x-1} = 9 \cdot \left(2^{x-1}\right)^2 > 9(x-1)^2 > (x-1)^2 - x.$$

This proves our claim.

**4.** Let $ABC$ be a right triangle with $\angle ACB = 90^\circ$ and let $D$ be a point on the side $AB$ different from both $A$ and $B$. Let points $E$ and $F$ be chosen on the sides $AC$ and $CB$ respectively, so that $AC \perp ED$ and $CB \perp FD$. Prove that $S_{ABC} \geq 4S_{EFD}$, where $S_{XYZ}$ denotes the area of the triangle $XYZ$.

**Solution.** Let $x = |FD|$, $y = |ED|$, $a = |BC|$ and $b = |AC|$. Since $ABC$ and $DBF$ are similar triangles and $|CF| = |ED| = y$ we get that

$$\frac{x}{b} = \frac{a-y}{a}.$$

Now using arithmetic-geometric mean inequality of the form

$$\frac{a-y+y}{2} \geq \sqrt{(a-y)y}$$

we get

$$S_{EFD} = \frac{xy}{2} = \frac{b}{2a}(a-y)y \leq \frac{b}{2a}\left(\frac{a-y+y}{2}\right)^2 = \frac{ab}{8} = \frac{S_{ABC}}{4},$$

which is what we wanted to prove.

**5.** A sequence $a_1, a_2, \dots$ of positive real numbers is called harmonic if for all $n > 0$, $a_{n+1}$ is the harmonic mean of $a_n$ and $a_{n+2}$, i.e. if for all $n > 0$ we have:

$$a_{n+1} = \frac{2}{\frac{1}{a_n} + \frac{1}{a_{n+2}}}.$$

Let $a_1, a_2, \dots$ be a harmonic sequence of positive real numbers with $a_1 = 1$ and $a_2 = \frac{1}{2}$. Find $a_{2015}$.

**Solution.** Let us define a sequence $b_1, b_2, \ldots$ with $b_n = \frac{1}{a_n}$. We know that for all $n \geq 1$, the sequence satisfies

$$\frac{1}{b_{n+1}} = \frac{2}{b_n + b_{n+2}}$$

which is equivalent to

$$b_{n+1} = \frac{b_n + b_{n+2}}{2}.$$

So, for all $n \geq 1$, the following holds

$$b_{n+2} = 2b_{n+1} - b_n.$$

We know that $b_1 = 1$ and $b_2 = 2$. Now, we can calculate

$$b_3 = 2b_2 - b_1 = 4 - 1 = 3$$

and

$$b_4 = 2b_3 - b_2 = 6 - 2 = 4.$$

The first terms of the sequence $b_1, b_2, \ldots$ are $1, 2, 3, 4, \ldots$.

Let us prove by induction that $b_n = n$ for $n \geq 5$. We assume the result for $1 \leq k \leq n$ and prove it for $n + 1$. We find

$$b_{n+1} = 2b_n - b_{n-1} = 2n - (n - 1) = n + 1.$$

Therefore, $b_n = n$ for all $n \geq 1$. This implies that $a_n = \frac{1}{n}$ for all $n \geq 1$. So, $a_{2015} = \frac{1}{2015}$.